Analysis and Optimization of Contact Ratio of Asymmetric Gears

Alexander L. Kapelevich and Yurii V. Shekhtman

Introduction
The contact ratio of spur gears is a critical parameter that affects gear drive performance. The influence of this parameter on the gear drive load capacity, efficiency, and noise and vibration is well known. There are studies (Refs. 1–3) dedicated to the analytical and experimental comparison of gears with low and high contact ratios. The dynamics and efficiency of high-contact-ratio asymmetric tooth gears were described in (Refs. 4–5).

These publications explore contact ratio using a very similar evaluation approach. The gears are designed traditionally, based on a preselected basic (or generating) rack. This makes the contact ratio dependent on the number of teeth of mating gears, basic rack addendum, and X-shifts. A contact ratio is considered nominal, as it is designed without influence of deflections under the operating load. Comparable gear sets with different contact ratios are identical in numbers of teeth, tooth size, and modules.

Such comparisons might have some theoretical value, but for practical gear design, equalizing some performance parameters in comparable gears is more important. For example, high-contact ratio gears provide load sharing between two or three pairs of teeth, increasing the load capacity. However, when they are compared with high-pressure angle and low-contact ratio gears (assuming identical numbers of teeth and tooth sizes), the mesh efficiency of high-contact ratio gears is significantly lower, because of their long tooth addendums and low pressure angle. Now a gear designer faces a dilemma: what is more important, high load capacity or high gear efficiency? Comparing gear sets with identical numbers of teeth and tooth size shows that it is impossible to simultaneously maximize both of these performance factors.

This article presents an analysis of asymmetric tooth gears considering the effective contact ratio that is also affected by bending and contact tooth deflections. The goal is to find an optimal solution for high performance gear drives, which would combine high load capacity and efficiency, as well as low transmission error (which affects gear noise and vibration).

Effective Contact Ratio and Transmission Error
The (trademarked) Direct Gear Design method (Ref.6) defines the nominal contact ratio for external gears as:

\[ \epsilon_n = \frac{x_2}{2\pi} \left( \tan \alpha_1 + \frac{\alpha_1 \tan \alpha_2}{1 + \alpha_1} \right) \]  

(1)

where:
- \( \epsilon_n \) = Operating pressure angle
- \( \alpha_1 \) and \( \alpha_2 \) = Outer diameter profile angles
- \( u = \frac{z_2}{z_1} \) = Gear ratio
- \( z_1 \) and \( z_2 \) = Number of teeth of mating pinion and gear

Effective contact ratio can be defined as the ratio of the tooth engagement angle to the angular pitch. The tooth engagement angle is a gear rotation angle from the start of the tooth engagement with the mating gear tooth to the end of the engagement. The effective contact ratio is:

\[ \epsilon_x = \frac{\varphi_1}{360/z_1} = \frac{\varphi_2}{360/z_2} \]  

(2)

where:
- \( \varphi_1 \) and \( \varphi_2 \) = Pinion and gear engagement angles
- \( 360/z_1 \) and \( 360/z_2 \) = Pinion and gear angular pitches

Transmission error is (Ref.7)

\[ TE = r_{n2} (\theta_2 - u \theta_1) \]  

(3)

where:
- \( \theta_1 \) and \( \theta_2 \) = Driving pinion and driven gear rotation angles
- \( r_{n2} \) = Driven gear base radius

Figure 1 Transmission error chart: \( \Delta \) = distance in microns between actual tooth contact point and ideal contact point.

A typical spur gear transmission error chart is shown (Fig. 1). The effective contact ratio and transmission error are influenced by manufacturing tolerances and operating conditions, including deflections under load, temperature, etc. of the gears and other gearbox components. In this article, only bending and contact tooth deflections are considered for the definition of the effective contact ratio and transmission error. Each angular position of the driven gear relative to the driving gear is iteratively defined by equalizing the sum of the tooth contact load moments of each gear to its applied torque. The related
tooth contact loads are also iteratively defined to conform to tooth bending and contact deflections, where the tooth bending deflection in each contact point is determined based on the FEA-calculated flexibility and the tooth contact deflection is calculated by the Hertz equation.

## Comparable Gear Analysis

Comparable gear set macro geometry is defined by the Direct Gear Design method (Ref. 6); it allows for having the drive flank nominal contact ratio as one of the gear design input parameters. The mating gears have optimized root fillets. The specific sliding velocities are equalized to maximize gear mesh efficiency, which for external spur gears is equal to (Ref. 7):

$$E = 100 \times \left( 1 - \frac{f}{2 \cos \alpha} \times \frac{H_2^2 + H_1^2}{H_1 + H_1} \right) \%$$  (4)

where:

- $f$ = Average friction coefficient
- $H_1$ = Specific sliding velocity at start of approach action
  $$H_1 = (1 + u) \times \cos \alpha \times \left( \tan \alpha_2 - \tan \alpha_1 \right)$$  (5)
- $H_2$ = Specific sliding velocity at end of recess action
  $$H_2 = (1 + u) \times \cos \alpha \times \left( \tan \alpha_2 - \tan \alpha_1 \right) / u$$  (6)

Maximum gear mesh efficiency is achieved when the specific sliding velocities $H_1 = H_2$ are equalized. Then maximum mesh efficiency for external spur gears can be defined from Equations 4–6 (considering also Eq. 1) as:

$$E = 100 \times \left( 1 - \frac{f \left( (1 + u) \times \cos \alpha \times \left( \tan \alpha_2 - \tan \alpha_1 \right) \right)}{2 \cos \alpha} \right) \%$$  (7)

All comparable gear sets are assumed to have identical maximized mesh efficiency $E$, average friction coefficient $f$, and gear ratio $u$. Then, according to Equation 7, the nominal contact ratio is inversely proportional to the pinion's number of teeth, as in:

$$\frac{E_u}{E_1} = \left( 1 - \frac{E}{100} \right) \frac{2u}{\sin (1 + u)} \text{ const.}$$  (8)

The criterion 8 is used to analyze parameters of external spur gear sets with asymmetric teeth. Comparable asymmetric tooth gear sets have different numbers of teeth and identical center distance $a_w$, gear ratio $u$, coast flank pressure angle $\alpha_{c_\infty}$, normal pinion and gear tooth tip thicknesses $t_{h_1}$ and $t_{h_2}$ that are required to avoid the harden through tooth tips for the carburized harden gears, average friction coefficient $f$ and gear mesh efficiency $E$, pinion and gear material properties, and equalized specific sliding velocities $H_1$ and $H_2$. The face widths $b_1$ and $b_2$ are defined to approximately equalize the pinion and gear tooth bending stresses considering the optimized root fillets.

If the center distance is identical for all gear sets, the operating modules are inversely proportional to the number of pinion teeth and defined as:

$$m = \frac{2a_w}{z_e(1 + u)}$$  (9)

The operating pitch diameter tooth thickness ratio:

$$TTR = \frac{S_{w1}}{P_w} = \frac{S_{w1}}{(S_{w1} + S_{w2})}.$$  (10)

where:

- $S_{w1}$ and $S_{w2}$ = Pinion and gear teeth thicknesses at the operating pitch diameters
- $P_w$ = Operating circular pitch

The operating pitch diameter tooth thickness ratio value is $TTR$ selected to provide equalized specific sliding velocities $H_1$ and $H_2$ and identical pinion and gear tooth tip thicknesses $t_{h_1}$ and $t_{h_2}$.

The maximized drive flank pressure angle $\alpha_{c_\infty}$ is defined to achieve minimal contact stress. It must also provide the nominal drive contact ratio $\alpha_{c_\infty}$ defined by Equation 1, the preselected values of the coast flank pressure angle $\alpha_{c_\infty}$, and pinion and gear tooth tip thicknesses $t_{h_1}$ and $t_{h_2}$.

The asymmetry factor is:

$$K = \cos \alpha_{c_\infty} / \cos \alpha_{c_\infty}$$  (11)

The bearing load is:

$$F = 2000T_1/d_{bl1}$$  (12)

where:

- $T_1$ = Pinion operating torque in Nm
- $d_{bl1}$ = Pinion drive base diameter in mm

Load sharing factor is:

$$L = F_{max}/F,$$  (13)

where:

- $F_{max}$ = Maximum contact load in the single tooth set contact

### Table 1

<table>
<thead>
<tr>
<th>Gear set comparison conditions</th>
<th>Center distance – 150 mm; Gear ratio – 2:1; Coast Pressure Angle – 15°; Pinion and Gear face widths – 35 mm and 30 mm; Tooth tip thickness – 0.38 mm; Average friction coefficient – 0.05; Gear mesh efficiency – 99%; Pinion and Gear material properties: Modules of elasticity – 207,000 MPa, Poisson ratio – 0.3; Pinion Torque – 1500 Nm; All gears have optimized tooth root fillets.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numbers of teeth</td>
<td>Pinion</td>
</tr>
<tr>
<td>Gear</td>
<td>28</td>
</tr>
<tr>
<td>Tooth Thickness Ratio</td>
<td>0.518</td>
</tr>
<tr>
<td>Drive Pressure Angle, °</td>
<td>42.0</td>
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<tr>
<td>Asymmetry Factor</td>
<td>1.300</td>
</tr>
<tr>
<td>Nominal Drive Contact Ratio</td>
<td>1.11</td>
</tr>
<tr>
<td>Effective Drive Contact Ratio</td>
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</tr>
<tr>
<td>Specific Sliding Velocities</td>
<td>0.279</td>
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<tr>
<td>Bearing Load, N</td>
<td>40368</td>
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<tr>
<td>Load Sharing Factor</td>
<td>1.0</td>
</tr>
<tr>
<td>Contact Ratio Type</td>
<td>Low</td>
</tr>
<tr>
<td>Selected Gear Sets</td>
<td>-</td>
</tr>
</tbody>
</table>
If the drive flank effective contact ratio $\varepsilon_{e_{dc}} < 2.0$, the load sharing factor $L = 1.0$. Similar to the effective contact ratio and transmission error, the load sharing factor is defined accounting only for the bending and contact tooth deflections.

- Table 1 presents gear parameters for several gear sets that are defined to satisfy pre-selected comparison conditions. The highlighted parameters for four gear sets are selected to define the transmission error under variable operating loads and to find a gear set with the optimal contact ratio.

- Gear set 1 has a 15-tooth pinion and 30-tooth gear with a low contact ratio ($\varepsilon_{e_{ad}} = 1.19$ and $\varepsilon_{e_{dc}} = 1.33$).

- Gear set 2 has a 19-tooth pinion and 38-tooth gear with a medium contact ratio ($\varepsilon_{e_{ad}} = 1.51$ and $\varepsilon_{e_{dc}} = 1.69$).

- Gear set 3 has a 23-tooth pinion and 46-tooth gear with a transitional contact ratio ($\varepsilon_{e_{ad}} = 1.83$ and $\varepsilon_{e_{dc}} = 2.04$). It is called transitional because it has a nominal contact ratio $< 2.0$ and an effective contact ratio under the given operating load $> 2.0$. Such gear sets under low load have one or two mating tooth pairs in contact. When the load is increased to its operating level and tooth deflections are increased, the gears are engaged in two or three mating tooth pairs in contact. These results in tooth load sharing and a single-tooth load reduction.

- Gear set 4 has a 27-tooth pinion and 54-tooth gear with a high contact ratio ($\varepsilon_{e_{ad}} = 2.15$ and $\varepsilon_{e_{dc}} = 2.40$).

The main gear parameters vs. pinion number of teeth charts are shown (Fig. 2).

The Figure 2 charts indicate that with increasing numbers of pinion teeth, the tooth thickness ratio TTR also increases slightly, the drive flank pressure angle $\alpha_{ad}$ lowers, but the nominal and effective contact ratios $\varepsilon_{ad}$ and $\varepsilon_{dc}$ grow. As a result of

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<table>
<thead>
<tr>
<th>Table 2 Results of selected gear set analysis under different driving torques</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gear set 1—low contact ratio: $\varepsilon_{e_{ad}} = 1.194$, $z_2 = 15$, $z_1 = 30$, $m = 6.676$ mm, $a_{pm} = 39.0^\circ$, $a_{ase} = 15.0^\circ$</td>
</tr>
<tr>
<td>Pinion Torque, Nm</td>
</tr>
<tr>
<td>Contact Stress, MPa</td>
</tr>
<tr>
<td>Pinion Bending Stress, MPa</td>
</tr>
<tr>
<td>Gear Bending Stress, MPa</td>
</tr>
<tr>
<td>Effective Drive Contact Ratio</td>
</tr>
<tr>
<td>Transmission Error, mm</td>
</tr>
<tr>
<td>Load Sharing Factor</td>
</tr>
</tbody>
</table>

Gear set 2—medium contact ratio: $\varepsilon_{e_{ad}} = 1.512$, $z_2 = 19$, $z_1 = 38$, $m = 5.263$ mm, $a_{pm} = 31.1^\circ$, $a_{ase} = 15.0^\circ$

Gear set 3—transitional contact ratio: $\varepsilon_{e_{ad}} = 1.831$, $z_2 = 23$, $z_1 = 46$, $m = 4.346$ mm, $a_{pm} = 25.8^\circ$, $a_{ase} = 15.0^\circ$

Gear set 4—high contact ratio: $\varepsilon_{e_{ad}} = 2.149$, $z_2 = 27$, $z_1 = 54$, $m = 3.704$ mm, $a_{pm} = 22.3^\circ$, $a_{ase} = 15.0^\circ$

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[www.geartechnology.com]
Figure 4  a) nominal and effective contact ratios; b) load sharing factor.

Figure 5  Pinion a) and gear b) bending stress charts.

Figure 6  a) contact stress; b) transmission error.
the drive flank pressure angle reduction, the bearing load $F$ is noticeably reduced, but the equalized specific sliding velocities $H_s$ and $H_t$ are increased because of the increased contact ratios.

Figure 3 shows the selected gear set meshes at the same scale; arrows indicate the driving pinion torque direction.

Results of the selected gear sets analysis under different driving torques are shown in the Table 2.

Main gear parameters vs. pinion operating torque for gear sets 1–4 are shown in Figs. 4–6.

Figures 7 and 8 present the transmission error charts of gear sets 1–4 at different driving torques.

The charts in Figure 6b clearly indicate that with increasing operating torque, the transmission error of gear sets 1, 2, and 4 increases as well. In gear set 3 the transmission also increases until the effective contact ratio exceeds 2.0 and the gear engagement is converted from 1–2 mating tooth pair contact to the 2–3 mating tooth pair contact. Then the transmission error of gear set 3 decreases slightly, stays flat, and then gradually increases. Within the operating torque range, gear set 3’s transmission error is the lowest in comparison to the other gear sets.

Summary

The article presents an analysis of nominal and effective contact ratios of several sets of spur asymmetric tooth gears with equal maximized gear mesh efficiencies but different numbers of teeth. This analysis has defined the main gear performance parameters, including tooth bending and contact stresses and transmission errors under variable operating load, accounting for bending and contact stress deflection. It demonstrated that transitional contact ratio gears appeared to be an optimal solution within the operating load range, providing minimal transmission error, bending stress that is lower than that of gear sets with medium and high contact ratio gears, and contact stress that is lower than that of gear sets with low and medium contact ratio gears. These transitional contact ratio gears with relatively constant transmission error within operating load range are potentially good for tooth flank microgeometry optimization for additional transmission error reduction. This analysis confirms the article (Ref.3) conclusion that gears with integer values for the contact ratio are inherently quiet, when the effective contact ratio is considered instead the nominal contact ratio.

The presented contact ratio analysis and optimization method is equally applicable and can be very beneficial for directly designed symmetric tooth gears.
References

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